Oscillations of complex networks

Xingang Wang,[1,2] Ying-Cheng Lai,[3] and Choy Heng Lai[1,2]

[1]*Department of Physics, National University of Singapore, Singapore 117542*
[2]*Beijing-Hong Kong-Singapore Joint Centre for Nonlinear & Complex Systems (Singapore),*
National University of Singapore, Kent Ridge, Singapore 119260
[3]*Department of Electrical Engineering, Department of Physics and Astronomy,*
Arizona State University, Tempe, Arizona 85287, USA

A complex network processing information or physical flows is usually characterized by a number of macroscopic quantities such as the diameter and the betweenness centrality. An issue of significant theoretical and practical interest is how such a network responds to sudden changes caused by attacks or disturbances. By introducing a model to address this issue, we find that, for a finite-capacity network, perturbations can cause the network to *oscillate* persistently in the sense that the characterizing quantities vary periodically or randomly with time. We provide a theoretical estimate of the critical capacity-parameter value for the onset of the network oscillation. The finding is expected to have broad implications as it suggests that complex networks may be structurally highly dynamic.

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The response of a complex network to sudden changes such as intentional attacks, random failures, or abnormal load increase, has been of great interest [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] since the discoveries of the small-world [11] and the scale-free [12] topologies. The issue is particularly relevant for scale-free networks that are characterized by a power-law degree distribution. For such a network, generically there exists a small set of nodes with degrees significantly higher than those of the rest of the nodes. A scale-free network is thus robust against random failures, but it is vulnerable to intentional attacks [1]. This is particularly so when dynamics on the network is taken into account, which can lead to catastrophic breakdown of the network via the cascading process [5, 8] even when the attack is on a single node. A basic assumption underlying the phenomenon of cascading breakdown is that a node fails if the load exceeds its capacity. As a result, the load of the failed node has to be transferred to other nodes, which causes more nodes to fail, and so on, leading to a cascade of failures that can eventually disintegrate the network.

There are situations in complex networks where overload does not necessarily lead to failures. For instance, in the Internet, when the number of information-carrying packets arriving at a node exceeds what it can handle, traffic congestion occurs. That is, overload of a node can lead to the waiting of packets but not to the failure of the node. As a result of the congestion, traffic detour becomes necessary in the sense that any optimal routes for new packets on the network try to avoid the congested nodes. This is equivalent to a change in the "weights" (to be defined more precisely below) of the congested nodes and, consequently, to changes in the macroscopic characterizing quantities of the network. This situation usually does not occur when the network is in a normal operational state, but it becomes likely when sudden disturbances, such as an attack or an abrupt large load increase, occur. A question is then whether the network can recover after a finite amount of time, in the sense that its characterizing quantities restore to their original values.

In this Letter, we study a class of weighted scale-free networks, incorporating a feasible traffic-flow protocol, to address the above question. In the absence of any perturbations, the network is assumed to operate in its "normal" state so that its macroscopic characterizing quantities are constants. We find that, after a large perturbation, the network can indeed recover but only for large node capacities. When the node capacities are not significantly higher than their loads in the normal state, a surprising phenomenon arises: the macroscopic quantities of the network are never able to return to their unperturbed values but instead, they exhibit persistent oscillations. In this sense we say the *network oscillates*. More remarkably, as the node capacities are decreased, both periodic and random oscillations can occur. The striking feature is that the oscillations, periodic or random, are caused solely by the interplay between the network topology and the traffic-flow protocol, regardless of the network parameters such as the degree distribution and the overall load fluctuations. For fixed network parameters, the oscillations exist despite of the explicit form of the local node dynamics, given the simple rule that it "holds" and causes the traffic to wait when overloaded. This may have wide implications to many traffic problems. For instance, it can provide an alternative explanation, from the dynamical point of view, for the recently observed random oscillations in real Internet traffic flow [13, 14] and give some insights to the self-similar oscillations of the traffic flux observed in WWW [15, 16].

We begin by constructing a scale-free network of N nodes using the standard growth and preferential-attachment mechanism [12]. We next define the node capacity by using the model in Ref. [8],

$$C_i = (1 + \alpha)L_i(0), \tag{1}$$

where $L_i(0)$ is the initial load on node i, which is approximately the load in a normal operational state (free of traffic congestion), and $\alpha > 0$ is the *capacity* parameter. The load L_i can be conveniently chosen to be the betweenness [17, 18], which is the total number of optimal paths [19] between all pairs of nodes passing through node i. To define an optimal

path at time t, say at this time the weights associated with node i and with node j are $w_i(t)$ and $w_j(t)$ (to be defined below according to the degree of traffic congestion), respectively, where there is a direct link l_{ij} between the two nodes. The weight of the link is then $d_{ij}(t) = [w_i(t) + w_j(t)]/2$. Given a pair of nodes, one packet generating and another receiving, the optimal path is the one which minimizes the sum of all weights d_{ij} of links that constitute the path. Finally, we define a traffic protocol on the network by assuming that, at each time step, one packet is to be communicated between any pair of nodes. There are thus $N(N-1)/2$ packets to be transported across the whole network at any time. When a packet is generated, its destination and the optimal path that the packet is going to travel toward it are determined.

In a computer or a communication network, a meaningful quantity to characterize a link is the time required to transfer a data packet through this link. When the traffic flow is free, it takes one time unit for a node to transport a packet. When congestion occurs, it may take a substantially long time for a packet to pass through a node and hence any link from this node. For instance, suppose at time t there are $J_i(t)$ packets at node i, where $J_i(t) > C_i$. Since the node can process C_i packets at any time, the waiting time for a packet at the end of the queue is $1 + \text{int}[J_i(t)/C_i]$, where $\text{int}[\cdot]$ is the integer part of the fraction in the square bracket. These considerations lead to the following definition of *instantaneous* weight for node i:

$$w_i(t) = 1 + \text{int}[\frac{J_i(t)}{C_i}], \text{ for } i = 1, \ldots, N, \quad (2)$$

from which the instantaneous weights for any link in the network and hence a set of instantaneous optimal paths can be calculated accordingly. For free traffic flow on the network, we have $J_i(t) < C_i$ and hence $w_i(t) = 1$ so that the network is non-weighted. In this case, the optimal path reduces to the shortest path.

The above model of traffic dynamics on a weighted network allows us to investigate the response of the network to perturbations in a systematic way. In particular, since the node capacities are the key to the occurrence of traffic congestion, it is meaningful to choose the capacity parameter α in Eq. (1) as a bifurcation parameter. To apply perturbation, we locate the node with the largest betweenness B_{max} in the network and generate a large number of packets, say ten times of B_{max}, at time $t = 0$. The network is then allowed to relax according to our model. Initially, because of the congestion at the largest-betweenness node caused by the perturbation, its weight assumes a large value. As a result, there is a high probability that the optimal paths originally passing through this node change routes. This can lead to a sudden increase in the network diameter, which is the average of all optimal paths. As time goes by, the congestion will cascade to other nodes which adopt the detoured optimal paths and, as a result, the diameter will increase very quickly and reaches its maximum at some instant. During this process the congestion situation is released at the attacking node while it get worse at nodes that paths detour to. After this, the network begins to "absorb" the congestions according to the load tolerance, i.e.
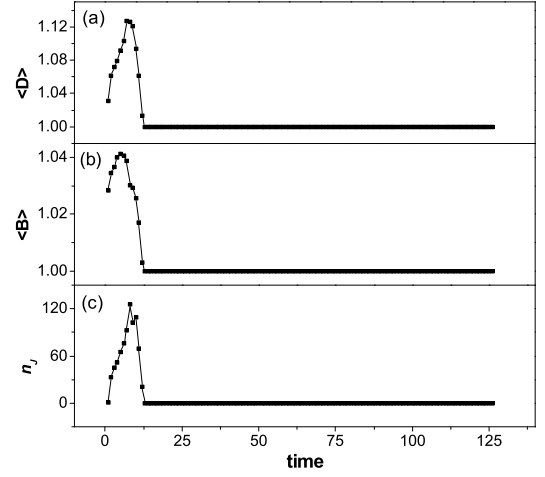

FIG. 1: For a scale-free network of 1000 nodes and for $\alpha = 0.4$, time evolutions of three macroscopic quantities: (a) the normalized network diameter, (b) the normalized betweenness centrality, and (c) the number of jammed nodes. The quantities return to their respective steady-state values after a brief transient.
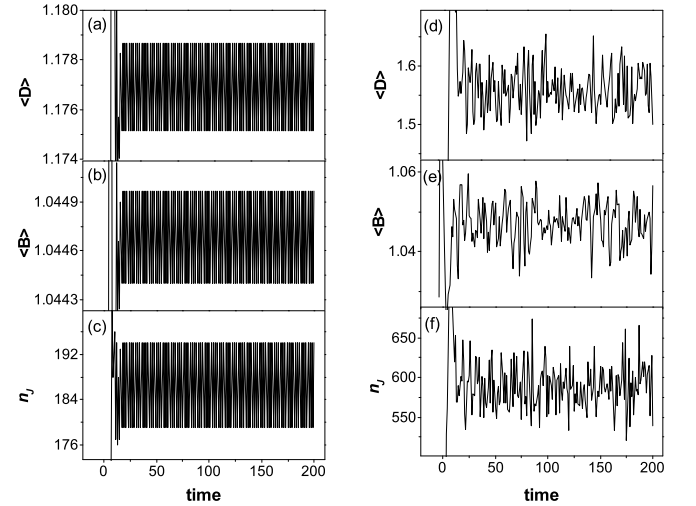

FIG. 2: (a-c) For the same scale-free network but for $\alpha = 0.31$, periodic oscillations of the normalized diameter (a), the normalized betweenness centrality (b), and the number of jammed nodes (c). (d-f) random oscillations of the same set of quantities for $\alpha = 0.2$.

the capacity parameter, and the recovery process starts. During this process the network congestion is gradually released and the diameter is expected to be decreasing. The same processes apply to other macroscopic characterizing quantities of the network, such as the betweenness centrality. However, due to the imbalance of load distribution and the high density of optimal paths, the final state where the system recovers to is highly unpredictable [23]. Therefore we are interested in whether these quantities can return to their "normal", or the steady-state values before the perturbation.

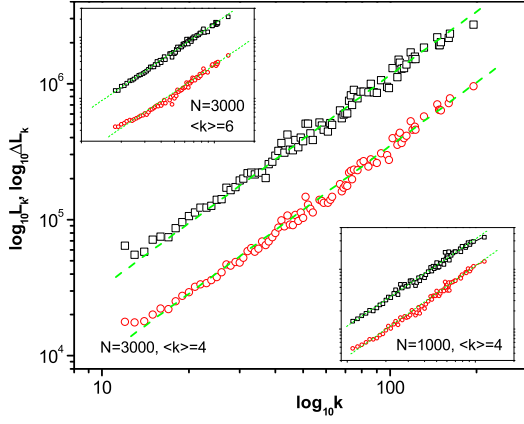For relatively large value of α, the ability of the network

FIG. 3: (Color online) For a scale-free network of $N = 3000$ and $\langle k \rangle = 4$, algebraic scaling of the initial load with the degree variable k (upper data set) and the scaling of the load change caused by unit weight change (lower data set). The parallelism of the two sets validates the use of Eq. (3). The insets show similar plots but for different network parameters. These results suggest that the critical value α_c for network oscillation is independent of the structural details of the network. The results are averaged over 50 realizations.

to process and transport packets is strong, so we expect the network to be able to relax to its unperturbed state. This is exemplified in Figs. 1(a-c), the time evolutions of three macroscopic quantities, the normalized diameter $\langle D \rangle$, the normalized betweenness centrality $\langle B \rangle$, and the number of jammed nodes n_J, respectively, of a scale-free network of 1000 nodes for $\alpha = 0.4$. [For this network the values of the diameter and of the betweenness centrality in the unperturbed state are $\langle D_0 \rangle \approx 5.18$ and $\langle B_0 \rangle \approx 2.35 \times 10^6$. The plotted quantities in (a) and (b) are normalized with respect to these "static" values.] We see that, after about 7 time steps, these quantities reach their maximum values and, after another about 5 steps, these quantities return to their respective unperturbed values. In this case, the large perturbation causes the network to oscillate but only for a transient time period. As the capacity parameter α is reduced, a remarkable phenomenon occurs: after an initial transient the network never returns to its steady-state but instead, it exhibits persistent oscillations. Figures 2(a-c) show periodic oscillations for $\alpha = 0.31$, where the legends are the same as for Figs. 1(a-c), respectively. The oscillations are in fact period-2 in that each macroscopic quantity can assume two distinct values, neither being the steady-state value, and the quantity alternates between the two values. For smaller value of α, random oscillations [20] occur, as shown in Figs. 2(d-f) for $\alpha = 0.2$.

The critical value α_c of the capacity parameter, below which persistent network oscillations can occur, can be estimated by noting that, for a given node j, the maximally possible increase in the load before traffic congestion occurs is $\alpha L_j(0)$. The weight-assignment rule in our traffic protocol, Eq. (2), stipulates that the most probable weight change be unity. Now regard α as a control parameter. For a *fixed* amount of change ΔL_j in the load, free flow of traffic is

guaranteed if $\alpha L_j(0) > \Delta L_j$ but traffic congestion occurs if $\alpha L_j(0) < \Delta L_j$. The critical value α_c is then given by

$$\alpha_c = \Delta L_j / L_j(0), \qquad (3)$$

which is independent of the degree variable k [8]. Since the load distribution with respect to k is algebraic [18], this suggests that, in order for Eq. (3) to be meaningful, ΔL_j must follow an algebraic scaling law with the *same* exponent. Since the amount of possible weight change is approximately fixed, the resulting load change is also fixed. To give an example, we consider a weighted scale-free network of parameters $N = 3000$ and $\langle k \rangle = 4$. Initially all nodes are assigned the same unit weight. The algebraic load distribution is shown in Fig. 3 (squares, the upper data set) on a logarithmic scale. The algebraic scaling exponent is about 1.5. Next we choose nodes of degree k and give them a sudden, unit increase in the weight. A recent work shows that for weighted scale-free networks, a weight increase of a node typically causes its load to decrease [21]. The load change ΔL as a function of k is shown in Fig. 3 (circles, the lower data set). We see that on the logarithmic scale, ΔL versus k is parallel to the initial load-degree distribution curve, justifying the use of Eq. (3). Numerically we obtain $\alpha_c^* \approx 0.37$. Since in a realistic situation there are more nodes with weights above the uniform background value of unity and since the amount of weight change can be more than unity, this value of α_c is only approximate. Indeed, direct numerical computations give $\alpha_c \approx 0.32$. The two estimates are nonetheless consistent. An interesting observation is that the value of α_c is insensitive to the network parameters like the network size and the degree distribution, as shown in the two insets in Fig. 3. In particular, for $N = 1000$ and $\langle k \rangle = 4$ (inset in the lower-right corner), we have $\alpha_c \approx 0.39$, while for $N = 3000$ and $\langle k \rangle = 6$ (upper-left corner), we obtain $\alpha_c \approx 0.40$. This phenomenon of network-parameter independency can be understood by noting that the load variation at a node caused by its weight change is mainly determined by the probability that optimal paths through this node appear or disappear, as a result of the weight change. This probability is independent of the network size and the degree of the node [21]. The value of α_c, of course, depends on the weight-assignment rule and thus on the traffic protocol.

Can oscillations be expected in realistic networks? To address this question, we test the stability of the Internet at the autonomous system level [22]. The network comprises 6474 nodes and 13895 links, the average diameter is $\langle D(0) \rangle \approx 4.71$, the largest value of the degree is 1460 and the load of this node is $L_j \approx 1.97 \times 10^8$. By setting $\alpha = 0.2$, we apply perturbation of strength $P = 10 \times L_j$ at the largest-degree node (to mimic an attack) and let the Internet evolve according to our traffic protocol. The time evolutions of the normalized diameter $\langle D \rangle$, of the normalized betweenness $\langle B \rangle$, and of the number of congested nodes n_J are shown in Fig. 4(a-c). Again, persistent oscillations are observed.

Since network oscillation is a result of the dynamical interplay between the complex topology and the traffic protocol, this phenomenon is expected to be common for complex systems. For simplicity in this paper we adopt the standard scale-free random network as the typical model to illustrate
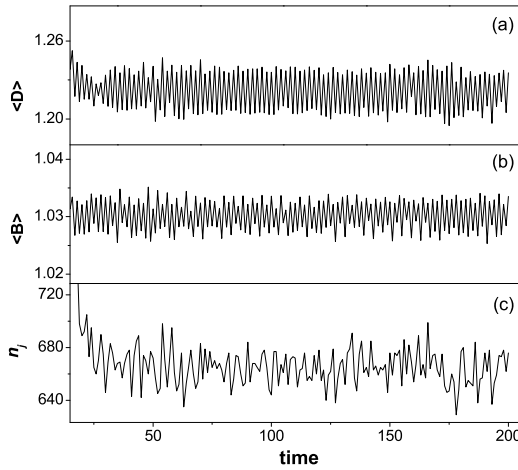
FIG. 4: For the Internet at the autonomous system level with capacity parameter $\alpha = 0.2$, evolutions of the normalized diameter (a), of the normalized betweenness centrality (b), and of the number of jammed nodes. Persistent oscillations of the Internet are observed.

the oscillation, but extensive numerical evidences have shown that this phenomenon is firmly established for complex networks independent of the network parameters like the network type, the degree distribution, the average degree, the network size, the perturbation size and position, and the specific queuing scheme of the local node dynamics. We note that although the oscillation phenomenon is robust, its time sequence is very sensitive to the network details. As a result, the transition from periodic oscillations to chaotic oscillations as the capacity decreases is non-smooth. Nonetheless, the trend from regular to complex oscillations is still clear. To save space, the details will be reported elsewhere [23].

To emphasize the nature of the oscillations, we had avoided

the fluctuations of capacity parameters and traffic load. While these fluctuations will enhance the oscillations, reflected as larger oscillation thresholds and larger oscillation amplitudes [23], the deterministic feature of the evolutionary network do exemplify the fact that oscillation is an intrinsic property of complex systems. This feature makes our model fundamentally different to the other traffic models in both the temporal- and spatial-domain [24]. Meanwhile, our model is also different to the congestion control models where chaotic flux oscillations are observed [13, 14]. In specific, we consider the competitions of simple node dynamics on complex topologies while the congestion models investigate the competitions of complicated node dynamics on simple topologies.

In summary, we have discovered that complex network of finite capacity can oscillate in the sense that its macroscopic quantities exhibit persistent periodic or random oscillations in response to external perturbations. While the study is arisen as a problem of network security, our findings may have broad implications to general traffic networks which, when applying the evolutionary weighted model, need further specifications. Whereas there can be all sorts of dynamical processes on a complex network, our findings suggest that there can be physically meaningful situations where the network itself is never static but highly dynamic. As a primary model for evolutionary weighted network, the oscillation of macroscopic quantities is only one aspect in describing the interplay between the topology and the local dynamics, further works in other aspects, such as adaptive networks where the network topology updates according to the local dynamics, will be very interesting.

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